

T-characteristic Function Based Multiobjective Linear Programming Problem With Fuzzy Coefficients Under Imprecise Environment (April 2015)

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Abstract—The primary objective of this paper is to find Pareto optimal solutions to multiobjective linear programming problem with fuzzy coefficients under imprecise environment. There exist several methods of Pareto optimization in literature. Recently Jimenez, and Bilbao showed that fuzzy efficient solution may not be Pareto optimal when one of the fuzzy goals is fully achieved. Next Wu, Liu, and Lur redefined membership function of fuzzy set theory. But we have observed that redefined membership function was not used during computation according to that proposed definition. And proper usage of membership or even redefined membership functions in fuzzy environment may suggest problem as infeasible. Moreover in decision making under imprecise environment,

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membership function is constructed so that higher membership grade gives better result for objective. And it is sometimes difficult or even impossible to construct fuzzy membership function or redefined membership function of imprecise objectives. And usage of membership function or redefined membership function in fuzzy environment gives at most one Pareto optimal solution to the problem. So in this paper, functions, namely T-characteristic functions, are used in place of usual membership functions or redefined membership functions in fuzzy environment. It results in more than one Pareto optimal solutions, and ultimately this feature helps decision maker. We have proposed a mathematical procedure to further illustrate usage of T-characteristic function in finding Pareto optimal solution to multiobjective linear programming problem with fuzzy coefficients under imprecise environment. Numerical examples illustrate our proposed procedure.

Index Terms—Fuzzy set theory, Linear programming, Optimization, Pareto optimization, T-characteristic function.

I. INTRODUCTION

WE consider the following multiobjective linear programming problem (MOLPP) with k objective functions $z_i(x) = c_i x$, $i = 1 \dots k$ as

$$\begin{aligned} \text{Min } z(x) &= (z_1(x), z_2(x), \dots, z_k(x))^T = (c_1 x, c_2 x, \dots, c_k x)^T \\ \text{subject to } x &\in X \end{aligned} \tag{A}$$

Here $X = \{x \in R^n : Ax \leq b, x \geq 0\}$, $b = (b_1, b_2, \dots, b_m) \in R^m$, and A is a $m \times n$ matrix. R denotes the set of reals. By assuming that the decision maker (DM) has imprecise aspiration levels for each of the objective functions $z_i(x) = c_i x$, $i = 1, \dots, k$ of model (A), researchers have proposed several methods in literature for characterizing Pareto optimal solutions to MOLPP (A). In one such approach, we use fuzzy set theory [7]. Here we convert MOLPP (A) into single objective optimization problem to obtain M-Pareto optimal solution, and then Pareto optimal solution by using the fuzziness of the DM's aspiration with respect to the goals of imprecise objective functions (constraints till they are symmetric).

One of the popular, well established methods to solve fuzzy MOLPP is two phase method. Here max-min operator is usually applied due to its easy computation but it does not guarantee to yield an M-Pareto optimal solution if more than one optimal solution exists.

Recently Jimenez, and Bilbao (2009) [9] observed that a fuzzy-efficient solution may not be a Pareto optimal solution in the case that one of fuzzy goals is fully achieved. And they proposed a procedure which extended the two-phase approach of Guu, Wu (1997, 1999) [14] and approach of Dubois, Fortemps (1999) [4] to attain a Pareto-optimal solution for model (A).

But according to Yan-Kuen Wu *et al* (2015) the proposed approach by Jimenez and Bilbao (2009) [9] cannot guarantee to be a general procedure to attain Pareto-optimal solution to model (A). In their proposed approach, Wu *et al* have modified membership function of fuzzy set theory and introduced redefined membership function. But that definition of redefined membership function $\mu_i(c_i x)$ and its usage in mathematical models as well as given numerical examples in their paper are not analogous to one another. From mathematical model or from

given numerical examples in that paper, it is clear that only one part of redefined membership function $\mu_i(c_i; x)$ where value of objective function does not exceed sum of goal, and tolerance (for minimization type of objective functions) is used. Yet these additional constraints were not added to set of existing constraints. Thus in problem formulation redefined membership function $\mu_i(c_i; x)$ is strictly monotone over entire domain where as it is not so in definition.

Actually Wu *et al* (2015) [17] removed upper bound of membership function only but there still lies a lower bound of it at zero. In those examples given by Wu *et al*, those additional constraints in definition of redefined membership function that objective value cannot exceed sum of goal, and tolerance (for minimization type of objective functions) have made no impact. But in many cases, if the lower bound of membership function remains intact, the problem may become infeasible. Point to be noted.

Deep analysis highlights the cause to lie in the definition of membership function of fuzzy set itself. Membership function of fuzzy set provides satisficing result when extreme ends of imprecise information can be quantified within the boundary of zero, and one. But it may not be always logically correct to quantify or measure every type of imprecise information with in some bounded subset of real line.

It is noted that an element of a fuzzy set can lie partly or never lie or must lie in that set. The idea of lying, lying partly or not lying is well quantified / measured by membership function of fuzzy set theory. But it does not suit in case of must lying (membership value always one), and never lying (membership value always zero). So usage of another function (in place of membership function or redefined membership function of fuzzy set) that best describes the

underlying issues well is now necessary.

Moreover in the models that closely describe as well as represent real life situations, the objective functions, and the constraints in MOLPP may involve parameters or coefficients many of whose possible values are usually assigned by the DM. In the conventional approach, those parameters are given fixed crisp values in an experimental and/or subjective manner through the understanding of DM. But in real life situations, however, it is natural to consider that the possible values of these parameters are often only ambiguously known to the DM. In this case, it becomes more appropriate to use fuzzy numbers as values of parameters. The resulting optimization problem involving fuzzy coefficients may be viewed as more realistic, and preferred version than the conventional one. Historically Tanaka, and Asai formulated the MOLPP with fuzzy parameters [7]. Sakawa (1986) used comprehensive, interactive technique to find fuzzy satisficing solution of MOLPP with fuzzy parameters [10]. Various other researchers have also worked on it in recent years.

We arrange this paper as follows: We first discuss the problem as well as existing methods to find solutions of MOLPP with fuzzy coefficients. The problems of using classical membership function or redefined membership function lead us to T-characteristic function [1], and consequently parametric T-characteristic function. Next T- α -Pareto optimal solution is defined analogous to existing M- α -Pareto optimal solution. Then we convert MOLPP with fuzzy coefficients under imprecise environment to a single objective linear programming problem by using proposed parametric T-characteristic functions, and discuss related lemmas. We integrate all those and develop an algorithm to find α -Pareto optimal solution to α -MOLPP for fixed α by using suitable parametric T-characteristic functions. First numerical example shows

advantages of using parametric T-characteristic function in place of membership or redefined membership function to solve α -MOLPP with fuzzy coefficients whereas second example is based on Eatman *et al* (1979), and shows how we generate different α -Pareto optimal solutions using different set of parametric T-characteristic functions under imprecise environment. Conclusions are drawn at last.

II. OVERVIEW OF MULTIOBJECTIVE LINEAR PROGRAMMING PROBLEM WITH FUZZY COEFFICIENTS

Let us consider the following MOLPP with fuzzy parameters having k objective functions $z_i(x) = C_i x$, $i = 1 \dots k$ as

$$\begin{aligned} \text{Min } z(x) &= (C_1 x, C_2 x, \dots, C_k x)^T \\ \text{Subject to } x &\in X(A, B). \end{aligned} \quad (1)$$

Here $X(A, B) = \{x \in R^n : A_j x \leq B_j, x \geq 0, j = 1 \dots m\}$, and $C_i = (C_{i1}, C_{i2} \dots C_{in})$ is an n-dimensional row vector of fuzzy coefficients involved in the ith objective function $C_i x$, $A_j = (A_{j1}, A_{j2} \dots A_{jn})$ is an n dimensional row vectors of fuzzy coefficients involved in the jth constraint $A_j x \leq B_j$, and B_j is a fuzzy coefficient of right hand side constant in the jth constraint $A_j x \leq B_j$. These coefficients reflect the DM's ambiguous understanding of the nature of the coefficients involved in the process of problem formulation, and hence all these coefficients $C_{i1}, C_{i2} \dots C_{in}, i = 1 \dots k; A_{j1}, A_{j2} \dots A_{jn}, j = 1, 2 \dots m$ and $B_j, j = 1 \dots m$ in the MOLPP with fuzzy coefficients are fuzzy numbers with membership functions $\mu_{C_{i1}}(C_{i1}), \mu_{C_{i2}}(C_{i2}) \dots \mu_{C_{in}}(C_{in}), i = 1 \dots k; \mu_{A_{j1}}(a_{j1}), \mu_{A_{j2}}(a_{j2}) \dots \mu_{A_{jn}}(a_{jn}), j = 1, 2 \dots m; \mu_{B_j}(b_j), j = 1 \dots m$. For the sake of simplicity alone, the

following notations may be used $c = (c_1 \dots c_k), a = (a_1 \dots a_m), b = (b_1 \dots b_m)$, and $C = (C_1 \dots C_k), A = (A_1 \dots A_m), B = (B_1 \dots B_m)$.

Here fuzzy numbers may be involved in both objective functions, and constraints in the MOLPP with fuzzy coefficients. And so the notion of Pareto optimality cannot be applied directly and it requires extension.

Again α -level set ($0 \leq \alpha \leq 1$) of the fuzzy numbers A_{jr}, B_j, C_{ir} is defined as crisp set $(A, B, C)_\alpha$ for which the degree of membership functions exceed the level α ($0 \leq \alpha \leq 1$) i.e.

$$(A, B, C)_\alpha = \{(a, b, c) : \mu_{A_{jr}}(a_{jr}) \geq \alpha, \mu_{B_j}(b_j) \geq \alpha, \mu_{C_{ir}}(c_{ir}) \geq \alpha; \\ i = 1 \dots k; j = 1 \dots m; r = 1 \dots n\}.$$

Suppose DM decides that degree of all membership functions of fuzzy numbers involved in the MOLPP with fuzzy parameters should be greater than or equal to some level α . So for such a given level α ($0 \leq \alpha \leq 1$), we select the coefficient vector $(a, b, c) \in (A, B, C)_\alpha$. Therefore the MOLPP with fuzzy parameters reduces to non-fuzzy MOLPP that depends on the coefficient vector $(a, b, c) \in (A, B, C)_\alpha$ in the form

$$\begin{aligned} \text{Min } z(x) &= (c_1 x, c_2 x \dots c_k x)^T \\ \text{subject to} & \\ x \in X(a, b) &= \{x \in R^n : a_j x \leq b_j, x \geq 0, j=1 \dots m\} \end{aligned} \tag{1A}$$

In fact infinite numbers of such MOLPP with fuzzy parameters exist for different values of coefficient vector $(a, b, c) \in (A, B, C)_\alpha$, and (a, b, c) are chosen arbitrarily in the sense that degrees of all the membership functions for the fuzzy numbers exceed the level α ($0 \leq \alpha \leq 1$). Actually the DM desires to choose $(a, b, c) \in (A, B, C)_\alpha$ so as to minimize the objective functions under

constraints for given $\alpha (0 \leq \alpha \leq 1)$. Hence it is natural to formulate non-fuzzy MOLPP from MOLPP with fuzzy coefficients, and a certain level α ($0 \leq \alpha \leq 1$) as follows:

$$\begin{aligned} \text{Min } z(x) &= (c_1x, c_2x, \dots, c_kx)^T \\ \text{subject to} & \\ x \in X(a,b) &= \{x \in R^n : a_jx \leq b_j, x \geq 0, j=1 \dots m\}, (a,b,c) \in (A,B,C)_\alpha. \end{aligned} \quad (2)$$

As per existing literature, it is α -MOLPP. The α -Pareto optimal solution, and then α -level optimal parameters can be obtained by regarding (x, a, b, c) as the decision variables in the α -MOLPP.

III. DEFINITIONS

In the case of decision making under fuzzy environment [13], one interesting, and useful property of membership function of fuzzy set is that higher value of membership function always gives better result for objective function. And there may exist not only membership function but also many such functions with this amazing characteristic. Hence one new function viz. T-characteristic function that handles underlying issues as well as preserve this amazing characteristic of membership function well, is defined.

T-characteristic function: Let S denotes the universal set. Then T-characteristic function of fuzzy subset $\tilde{A}$ of S is denoted by $T_{\tilde{A}}$, and is defined as $T_{\tilde{A}} : S \rightarrow R$. It assigns a real number $T_{\tilde{A}}(x)$ to each element $x \in S$, where $T_{\tilde{A}}(x)$ represents the grade of membership of $x \in S$ in $\tilde{A}$. Here R denotes the set of real numbers.

Parametric T-characteristic function: Let S denotes the universal set. Then parametric T-characteristic function of fuzzy subset $\tilde{A}$ of S with parameter $\alpha (0 \leq \alpha \leq 1)$ is denoted by $T_{\tilde{A}}(x, \alpha)$,

and is defined as $T_{\tilde{A}} : S \rightarrow R$. For each $\alpha (0 \leq \alpha \leq 1)$, it assigns a real number $T_{\tilde{A}}(x, \alpha)$ to each element $x \in S$, where $T_{\tilde{A}}(x, \alpha)$ represents the grade of membership of $x \in S$ in $\tilde{A}$.

Next, on the basis of α -level sets of fuzzy numbers, the concept of α -Pareto optimal solution to the α -MOLPP (2) as a natural extension of the Pareto optimality concept for the MOLPP (A) is given as follows:

α -Pareto optimal solution: A point $x^* \in X(a^*, b^*)$ is said to be a α -Pareto optimal solution to a α -MOLPP if and only if there does not exist another $x \in X(a, b)$, and $(a, b, c) \in (A, B, C)_\alpha$ such that $c_i x \leq c_i x^*, i = 1, \dots, k$ with strict inequality holding for at least one i . The corresponding values of the parameters (a^*, b^*, c^*) are called α -level optimal parameters.

Based on T-characteristic function, we now introduce the concept of T- α -Pareto optimal solution which is defined in terms of T-characteristic function instead of objective function.

T- α -Pareto optimal solution: A point $x^* \in X(a^*, b^*)$ is said to be a T- α -Pareto optimal solution to an α -MOLPP if and only if there does not exist another $x \in X(a, b)$, and $(a, b, c) \in (A, B, C)_\alpha$ such that $T_i(c_i x^*, \alpha) \leq T_i(c_i x, \alpha)$ for all $i = 1 \dots k$, $i \neq j$, and $T_j(c_j x^*, \alpha) < T_j(c_j x, \alpha)$ for at least one j for some fixed $\alpha (0 \leq \alpha \leq 1)$. The corresponding values of the parameters (a^*, b^*, c^*) are called α -level optimal parameters for same $\alpha (0 \leq \alpha \leq 1)$.

E.g. in MOLPP with fuzzy coefficients under imprecise environment, in case of maximization type of objective function z_1 ; one of parametric T-characteristic functions may be z_1 itself. And in case of minimization type of objective function z_2 ; one of parametric T-characteristic functions may be $(-)$ z_2 . They satisfy the same strictly monotonic property that higher value of

membership function always gives better result for objective function. And these may be used to convert a MOLPP with fuzzy coefficients or α -MOLPP into single objective optimization problem, and obtain T- α -Pareto optimal solution and hence α -Pareto optimal solution for given α ($0 \leq \alpha \leq 1$).

We note that the concept of T- α -Pareto optimal solution defined in terms of T-characteristic function of fuzzy set is a natural extension of M- α -Pareto optimal solution defined in terms of membership function of fuzzy set which is another natural extension of α -Pareto optimal solution defined in terms of objective functions.

IV. OBTAINING T- α -PARETO OPTIMAL SOLUTION AND α -PARETO OPTIMAL SOLUTION TO MOLPP WITH FUZZY COEFFICIENTS

Within the scope of the multiobjective decision making theory, the Pareto-optimality of solution is a necessary condition in order to guarantee the rationality of a decision. In a minimization problem, a fuzzy goal of DM may be to achieve “substantially less” than some value ξ . This type of statement can be quantified by eliciting corresponding suitable parametric T-characteristic functions. In some cases such description about goal may be completely unavailable. Then other techniques may also be employed to construct parametric T-characteristic functions of objective functions under fuzzy environment.

Using the method as discussed by Bellman, and Zadeh, for fixed α , ($0 \leq \alpha \leq 1$), (2) is equivalent to

$$\begin{aligned}
 & \max (T_1(c_1x), T_2(c_2x) \dots T_k(c_kx))^T \\
 & \text{subject to} \\
 & x \in X(a,b) = \{x \in R^n : a_j x \leq b_j, x \geq 0, j = 1 \dots m\}, (a,b,c) \in (A,B,C)_\alpha.
 \end{aligned} \tag{2A}$$

And this model (2A) is equivalent to

$$\begin{aligned}
 & \text{Min } \{(T_1(c_1x))^{-1}, (T_2(c_2x))^{-1} \dots (T_k(c_kx))^{-1}\}^T \\
 & \text{subject to} \\
 & x \in X(a,b) = \{x \in R^n : a_j x \leq b_j, x \geq 0, j = 1 \dots m\}, (a,b,c) \in (A,B,C)_\alpha.
 \end{aligned} \tag{2B}$$

Using the min-max method by Bowman (1976) [16], problem (2B) may be rewritten as

$$\begin{aligned}
 & \text{Min } \max_i \{(T_i(c_i x))^{-1}\}, i = 1, 2 \dots k \\
 & \text{subject to} \\
 & x \in X(a,b) = \{x \in R^n : a_j x \leq b_j, x \geq 0, j = 1 \dots m\}, (a,b,c) \in (A,B,C)_\alpha.
 \end{aligned} \tag{2C}$$

Let $v = \max_i \{(T_i(c_i x))^{-1}\}, i = 1, 2 \dots k$. Then model (2C) reduces to

$$\begin{aligned}
 & \min v \\
 & \text{subject to} \\
 & (T_i(c_i x))^{-1} \leq v, i = 1, 2 \dots k, \\
 & x \in X(a,b) = \{x \in R^n : a_j x \leq b_j, x \geq 0, j = 1 \dots m\}, (a,b,c) \in (A,B,C)_\alpha.
 \end{aligned} \tag{2D}$$

If we put $v = 1/\lambda$, then for same fixed α ($0 \leq \alpha \leq 1$), it further reduces to

$$\begin{aligned}
 & \max \lambda \\
 & \text{subject to} \\
 & T_i(c_i x) \geq \lambda, i = 1, 2 \dots k, \\
 & x \in X(a,b) = \{x \in R^n : a_j x \leq b_j, x \geq 0, j = 1 \dots m\}, (a,b,c) \in (A,B,C)_\alpha.
 \end{aligned} \tag{3}$$

The relationships between the T- α -Pareto optimal solution x^* of the problem (3), and the α -

Pareto optimal solution to the generalized MOLPP with fuzzy coefficients for fixed α i.e. α -MOLPP (2) can be characterized by following lemmas.

Lemma 1: If $x^* \in X(a_\alpha^L, b_\alpha^R)$ is unique T- α -Pareto optimal solution to the max-min problem (3), then x^* is α -Pareto optimal solution to the generalized α -MOLPP (2) for fixed α ($0 \leq \alpha \leq 1$).

Proof: Let $x^* \in X(a_\alpha^L, b_\alpha^R)$ is unique T- α -Pareto optimal solution to problem (3) for fixed α ($0 \leq \alpha \leq 1$). If possible let x^* is not a α -Pareto optimal solution to (2). Then there exist at least one $y \in X(a_\alpha^L, b_\alpha^R)$, and $(a, b, c) \in (A, B, C)_\alpha$ such that $c_i y \leq c_i x^*$ for all i , $i \neq j$, and $c_j y < c_j x^*$ for at least one j . Since each $T_i(c_i x)$, $i = 1, \dots, k$ is strictly monotonic decreasing function for each objective functions $z_i = c_i x$, $i = 1, 2, \dots, k$ of minimizing type, we get by using this relation $T_i(c_i x^*) \leq T_i(c_i y) \forall i = 1, 2, \dots, k, i \neq j$ and $T_j(c_j x^*) < T_j(c_j y)$. This gives $y \in X(a_\alpha^L, b_\alpha^R)$, and $(a, b, c) \in (A, B, C)_\alpha$ as an optimal solution to (3). It contradicts that $x^* \in X(a_\alpha^L, b_\alpha^R)$ is unique optimal solution to (3). Therefore our assumption is wrong. Hence $x^* \in X(a_\alpha^L, b_\alpha^R)$ is α -Pareto optimal solution to α -MOLPP (2) for same α ($0 \leq \alpha \leq 1$) ■

Lemma 2: If $x^* \in X(a_\alpha^L, b_\alpha^R)$ is α -Pareto optimal solution to the generalized α -MOLPP (2), then x^* is a T- α -Pareto optimal solution to the problem (3) for some T_i , $i = 1 \dots k$ for fixed α ($0 \leq \alpha \leq 1$).

Proof: Let $x^* \in X(a_\alpha^L, b_\alpha^R)$ is α -Pareto optimal solution to MOLPP (2) for some α ($0 \leq \alpha \leq 1$). If possible let $x^* \in X(a_\alpha^L, b_\alpha^R)$ is not a T- α -Pareto optimum solution to the problem (3) for any T_i , $i = 1, 2, \dots, k$ for α ($0 \leq \alpha \leq 1$). Then for T_i , $i = 1 \dots k$, there exists $x \in X(a_\alpha^L, b_\alpha^R)$, and $(a, b, c) \in (A, B, C)_\alpha$,

we have $T_j(c_j x) > T_j(c_j x^*) = \lambda(x^*)$, for some j , and $T_i(c_i x) \geq T_i(c_i x^*) = \lambda(x^*)$, $i = 1 \dots k, i \neq j$. But each T_i is strictly decreasing function for $i = 1 \dots k$. Hence we get $c_j x < c_j x^*$ for some j , and $c_i x \leq c_i x^*, i = 1, 2, \dots, k, i \neq j$. It contradicts that x^* is α -Pareto optimal solution to the α -MOLPP (2) for α ($0 \leq \alpha \leq 1$). Therefore our assumption is wrong. Hence x^* is an optimal solution to the problem (3) for $T_i, i = 1 \dots k$ for same α ($0 \leq \alpha \leq 1$) ■

From lemma 1, lemma 2, it is clear if the uniqueness of the optimal solution $x^* \in X(a_\alpha^L, b_\alpha^R)$, and $(a, b, c) \in (A, B, C)_\alpha$ for the max-min problem (3) is not guaranteed, it is necessary to perform α -Pareto optimality test of x^* . And the α -Pareto optimality test for x^* can be performed by solving the following linear programming problem with the decision variables $x = (x_1, x_2 \dots x_n)^T$ and $\varepsilon = (\varepsilon_1, \varepsilon_2 \dots \varepsilon_k)^T$.

$$\begin{aligned} & \max \sum_{i=1}^k \varepsilon_i \\ & \text{Subject to} \\ & T_i(c_i x) - \varepsilon_i \geq T_i(c_i x^*), i = 1 \dots k, \\ & x \in X(a_\alpha^L, b_\alpha^R), (a, b, c) \in (A, B, C)_\alpha, \varepsilon_i \geq 0, i = 1 \dots k \end{aligned} \tag{4}$$

Lemma 3: Let $\bar{x}$ and $\bar{\varepsilon}$ be optimal solution to (4) for some α ($0 \leq \alpha \leq 1$). Then

- 1) If all $\bar{\varepsilon}_i = 0$, then x^* is α -Pareto optimal solution of the α -MOLPP.
- 2) If at least one $\bar{\varepsilon}_i > 0$, then x^* is not α -Pareto optimal solution of the α -MOLPP.

Instead of x^* , $\bar{x}$ is α -Pareto optimal solution to the α -MOLPP.

Proof of similar lemma is recorded in literature. It may be noted that several authors have proposed different other procedures to test α -Pareto optimality and can be used in place of (4).

V. PROCEDURE TO SOLVE MULTIOBJECTIVE LINEAR PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS UNDER IMPRECISE ENVIRONMENT

The above ideas can be further integrated into a general framework and develop an algorithm to obtain α -Pareto-optimal solution to α -MOLPP with fuzzy coefficients for given α ($0 \leq \alpha \leq 1$), which is also T- α -Pareto optimal solution. The steps of the proposed algorithm are as follows:

- 1) Elicit suitable parametric T-characteristic function $T_i(c_i, x)$ for each imprecise objective function in such a way that higher value of parametric T-characteristic function gives better value of objective function. Well-posed parametric T-characteristic function preserves this characteristic and gives finite value.
- 2) DM selects initial value of the parameter α ($0 \leq \alpha \leq 1$). Otherwise, some suitable value of α can be taken.
- 3) Construct the max-min problem as (3) for chosen α ($0 \leq \alpha \leq 1$).
- 4) For this fixed α ($0 \leq \alpha \leq 1$), solve that max-min problem, and obtain the corresponding T- α -Pareto optimal solution, say x^* .
- 5) To test whether this T- α -Pareto optimal solution x^* to (3) is also α -Pareto optimal solution to (2), solve (4).
- 6) Let $\bar{x}$ and $\bar{\varepsilon}$ be optimal solution of (4) in step 5. Then two cases may arise:
 - a) If $\bar{\varepsilon}_i = 0 \forall i$, then x^* is α -Pareto optimal solution to α -MOLPP.

- b) If $\bar{\varepsilon}_j > 0$ for at least one j , then x^* is not α -Pareto optimal solution to α -MOLPP. Instead of x^* , $\bar{x}$ is α -Pareto optimal solution to the α -MOLPP.
- 7) This solution is T- α -Pareto optimal solution as well as α -Pareto optimal solution for fixed $\alpha(0 \leq \alpha \leq 1)$ to the α -MOLPP. And the algorithm is complete.

VI. NUMERICAL EXAMPLES

A. Numerical Example 1.

Here we consider an imprecise multiobjective linear programming problem with fuzzy coefficients as follows:

$$\begin{aligned}
 & \text{imprecise max } z_1 = 0.875x_1 - C_{12}x_2 \\
 & \text{imprecise min } z_2 = C_{21}x_1 + x_2 \\
 & \text{imprecise max } z_3 = C_{31}x_1 - 2x_2 \\
 & \text{subject to} \\
 & 5x_1 + 7x_2 \leq 12, \quad 9x_1 + x_2 \leq 10, \quad -5x_1 + 3x_2 \leq 3, \quad x_1 + x_2 \geq 1, \\
 & x_1 \geq 0, \quad x_2 \geq 0.
 \end{aligned}$$

For the sake of simplicity alone, we denote

$$X = \{(x_1, x_2) : 5x_1 + 7x_2 \leq 12, \quad 9x_1 + x_2 \leq 10, \quad -5x_1 + 3x_2 \leq 3, \quad x_1 + x_2 \geq 1, \quad x_1 \geq 0, \quad x_2 \geq 0\}$$

Here the coefficients C_{12}, C_{21}, C_{31} are fuzzy numbers with membership functions given by

$$\begin{aligned}
 \mu_{C_{12}}(C_{12}) &= \max(1 - |C_{12} - 1|, 0), \\
 \mu_{C_{21}}(C_{21}) &= \max(1 - |C_{21} - 5|, 0), \\
 \mu_{C_{31}}(C_{31}) &= \max(1 - 2|C_{31} - 6|, 0).
 \end{aligned}$$

Suppose that the DM choses value of the parameter as $\alpha = 0.7$. Then for $\alpha = 0.7$, the fuzzy coefficients C_{12}, C_{21}, C_{31} are triangular fuzzy numbers as

$C_{12}=[0.7, 1.3]$, $C_{21}=[4.7, 5.3]$, $C_{31}=[5.85, 6.15]$. We first solve this problem by using classical membership function of fuzzy set theory.

1) Imprecise α -MOLPP with first set of parametric T-characteristic functions analogous to usual membership functions:

To construct membership functions for each of the imprecise objectives, we find individual maximum, and minimum of each of the objective functions (properly using 0.7-cut of triangular fuzzy coefficients) under the given constraints as follows:

TABLE I
INDIVIDUAL MAXIMUM, AND MINIMUM

Objective Functions	Individual Maximum	Individual Minimum
$z_1(x) = 0.875x_1 - 1.3x_2$	0.97	- 1.7
$z_2(x) = 4.7x_1 + x_2$	5.7	1
$z_3(x) = 6.15x_1 - 2x_2$	6.8	-2

Next suppose that the DM specifies goal, and tolerance for each of the imprecise objective functions as follows:

TABLE II
GOALS, AND TOLERANCES OF OBJECTIVE FUNCTIONS

Objective Functions	Goals	Goal plus Tolerance
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		Values
$z_1(x)$	0.9	- 1
$z_2(x)$	1.5	3
$z_3(x)$	6.5	3

Based on the assigned goals, and tolerances, the membership functions of each of the imprecise objective functions may be formulated as follows:

$$T_1(c_1x) = \mu_1(c_1x) = \begin{cases} 1 & , \text{if } 0.875x_1 - 1.3x_2 \geq 0.9 \\ \frac{0.875x_1 - 1.3x_2 + 1}{1.9} & , \text{if } -1 \leq 0.875x_1 - 1.3x_2 \leq 0.9 \\ 0 & , \text{if } 0.875x_1 - 1.3x_2 \leq -1 \end{cases}$$

$$T_2(c_2x) = \mu_2(c_2x) = \begin{cases} 1 & , \text{if } 4.7x_1 + x_2 \leq 1.5 \\ \frac{3 - (4.7x_1 + x_2)}{1.5} & , \text{if } 1.5 \leq 4.7x_1 + x_2 \leq 3 \\ 0 & , \text{if } 4.7x_1 + x_2 \geq 3 \end{cases}$$

$$T_3(c_3x) = \mu_3(c_3x) = \begin{cases} 1 & , \text{if } 6.15x_1 - 2x_2 \geq 6.5 \\ \frac{6.15x_1 - 2x_2 - 3}{3.5} & , \text{if } 3 \leq 6.15x_1 - 2x_2 \leq 6.5 \\ 0 & , \text{if } 6.15x_1 - 2x_2 \leq 3 \end{cases}$$

If the DM does not specify goals, and tolerances, individual maximum, and minimum of objectives may be used to construct membership functions.

Then imprecise MOLPP with fuzzy coefficients becomes a single objective linear programming problem as follows:

$$\begin{aligned}
 & \max \quad \lambda \\
 & \text{subject to} \\
 & \quad \frac{0.875x_1 - 1.3x_2 + 1}{1.9} \geq \lambda, \\
 & \quad \frac{3 - (4.7x_1 + x_2)}{1.5} \geq \lambda, \\
 & \quad \frac{6.15x_1 - 2x_2 - 3}{3.5} \geq \lambda, \\
 & \quad 5x_1 + 7x_2 \leq 12, \quad x_1 + x_2 \geq 1, \\
 & \quad 9x_1 + x_2 \leq 10, \quad -5x_1 + 3x_2 \leq 3, \\
 & \quad 0 \leq \lambda \leq 1, \quad x_1, x_2 \geq 0.
 \end{aligned}$$

Using Lingo 15.0.32, we find that this problem has no feasible solution.

2) *Imprecise α -MOLPP with second set of parametric T -characteristic function analogous to redefined membership functions:*

Next we construct redefined membership functions $\mu^R_i(c_i x), i=1,2,3$ (as proposed by Wu *et al* [17]) based on the goal, and tolerance (assigned by DM) of each of the imprecise objectives as follows:

$$\begin{aligned}
 T_1(c_1 x) = \mu^R_1(c_1 x) &= \begin{cases} \frac{0.875x_1 - 1.3x_2 + 1}{1.9}, & \text{if } 0.875x_1 - 1.3x_2 \geq -1 \\ 0 & , \text{if } 0.875x_1 - 1.3x_2 \leq -1 \end{cases} \\
 T_2(c_2 x) = \mu^R_2(c_2 x) &= \begin{cases} \frac{3 - (4.7x_1 + x_2)}{1.5}, & \text{if } 4.7x_1 + x_2 \leq 3 \\ 0 & , \text{if } 4.7x_1 + x_2 \geq 3 \end{cases} \\
 T_3(c_3 x) = \mu^R_3(c_3 x) &= \begin{cases} \frac{6.15x_1 - 2x_2 - 3}{3.5}, & \text{if } 6.15x_1 - 2x_2 \geq 3 \\ 0 & , \text{if } 6.15x_1 - 2x_2 \leq 3 \end{cases}
 \end{aligned}$$

Based on these redefined membership function, we convert imprecise MOLPP with fuzzy coefficients to a single objective linear programming problem [4] as follows:

$$\begin{aligned}
 & \max \lambda \\
 & \text{subject to} \\
 & \frac{0.875x_1 - 1.3x_2 + 1}{1.9} \geq \lambda, \\
 & \frac{3 - (4.7x_1 + x_2)}{1.5} \geq \lambda, \\
 & \frac{6.15x_1 - 2x_2 - 3}{3.5} \geq \lambda, \\
 & 5x_1 + 7x_2 \leq 12, \quad 9x_1 + x_2 \leq 10, \quad -5x_1 + 3x_2 \leq 3, \quad x_1 + x_2 \geq 1, \\
 & \lambda, x_1, x_2 \geq 0.
 \end{aligned}$$

Using Lingo 15.0.32 to solve this problem, we find that the problem has no feasible solution.

3) Imprecise α -MOLPP with third set of parametric T-characteristic functions:

Next we introduce another set of parametric T-characteristic functions for each of the imprecise objective functions such that each of the parametric T-characteristic functions are strictly monotonic over entire domain and higher membership grade of parametric T-characteristic functions give better result for objectives.

Based on goals, and tolerances of imprecise objective functions, we define parametric T-characteristic functions for each of the imprecise objective functions as

$$T_1(c_1x) = \frac{0.875x_1 - 1.3x_2 + 1}{1.9}, \forall (x_1, x_2) \in X, \quad T_2(c_2x) = \frac{3 - (4.7x_1 + x_2)}{1.5}, \forall (x_1, x_2) \in X, \quad T_3(c_3x) = \frac{6.15x_1 - 2x_2 - 3}{3.5}, \forall (x_1, x_2) \in X.$$

Here membership grade between zero, and one represents partly presence. Membership grade more than one represents must presence whereas membership grade less than zero represents never presence. Using these parametric T-characteristic functions, we convert imprecise MOLPP with fuzzy coefficients to single objective linear programming problem as follows:

$$\begin{aligned} & \max \quad \lambda \\ & \text{subject to} \\ & \frac{0.875x_1 - 1.3x_2 + 1}{1.9} \geq \lambda, \frac{3 - (4.7x_1 + x_2)}{1.5} \geq \lambda, \frac{6.15x_1 - 2x_2 - 3}{3.5} \geq \lambda, \\ & 5x_1 + 7x_2 \leq 12, 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, x_1 + x_2 \geq 1, x_1, x_2 \geq 0, \lambda \text{ is unrestricted in sign.} \end{aligned}$$

Using Lingo 15.0.32, we obtain T- α -Pareto optimal solution as

$$\begin{aligned} \lambda^* &= -0.0873883, x_1^* = 0.5759682, x_2^* = 0.4240318, z_1(x^*) = -0.04726912, z_2(x^*) = 3.131082, \\ z_3(x^*) &= 2.694141, T_1(z_1(x^*)) = 0.5014373, T_2(z_2(x^*)) = -0.08738828, T_3(z_3(x^*)) = -0.08738828. \end{aligned}$$

Then we test α -Pareto optimality of T- α -Pareto optimal solution by solving the following:

$$\begin{aligned} & \max \quad \sum_{i=1}^3 \varepsilon_i \\ & \text{subject to} \\ & \frac{(0.875x_1 - 1.3x_2) + 1}{1.9} - \varepsilon_1 \geq 0.5014373, \\ & \frac{3 - (4.7x_1 + x_2)}{1.5} - \varepsilon_2 \geq -0.08738828, \\ & \frac{(6.15x_1 - 2x_2) - 3}{3.5} - \varepsilon_3 \geq -0.08738828, \\ & 5x_1 + 7x_2 \leq 12, 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, x_1 + x_2 \geq 1, \\ & x_1, x_2, \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0. \end{aligned}$$

Using Lingo 15.0.32, we obtain α -Pareto optimal solution $\bar{x}$ and $\bar{\varepsilon}$ as follows:

$$\begin{aligned} \bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 &= 0, \bar{x}_1 = 0.5759682, \bar{x}_2 = 0.4240318, z_1(\bar{x}) = -0.04726912, z_2(\bar{x}) = 3.131082, z_3(\bar{x}) = 2.694141, \\ T_1(z_1(\bar{x})) &= 0.5014373, T_2(z_2(\bar{x})) = -0.08738828, T_3(z_3(\bar{x})) = -0.08738828. \end{aligned}$$

Here all $\bar{\varepsilon}_i = 0 \forall i = 1, 2, 3$. Hence the solution obtained in previous stage is itself α -Pareto optimal solution to α -MOLPP.

4) Imprecise α -MOLPP with fourth set of parametric T-characteristic functions:

Again we may construct parametric T-characteristic functions for each of the imprecise

objective functions as follows:

$$T_1(c_1x) = 0.875x_1 - 1.3x_2, \forall (x_1, x_2) \in X, T_2(c_2x) = -(4.7x_1 + x_2), \forall (x_1, x_2) \in X, T_3(c_3x) = 6.15x_1 - 2x_2, \forall (x_1, x_2) \in X.$$

Here the parametric T-characteristic functions are again strictly monotonic over entire domain, and higher membership grade of parametric T-characteristic functions give better result for objectives. But membership grade zero or one of parametric T-characteristic functions do not represent grade of membership in usual fuzzy sense. Precisely we have picked up the only necessary property (strict monotonicity over entire domain) of membership function to construct our proposed parametric T-characteristic functions.

Based on these parametric T-characteristic functions, we convert the α -MOLPP into single objective linear programming problem [5] as follows:

$$\begin{aligned} & \max \lambda \\ & \text{subject to} \\ & 0.875x_1 - 1.3x_2 \geq \lambda, -(4.7x_1 + x_2) \geq \lambda, \\ & 6.15x_1 - 2x_2 \geq \lambda, 5x_1 + 7x_2 \leq 12, \\ & 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, \\ & x_1 + x_2 \geq 1, x_1, x_2 \geq 0, \lambda \text{ is unrestricted in sign.} \end{aligned}$$

Using Lingo 15.0.32, we obtain the T- α -Pareto optimal solution as

$$\lambda^* = -1.312236, x_1^* = 0.08438819, x_2^* = 0.9156118, z_1(x^*) = -1.116456, z_2(x^*) = 1.312236, z_3(x^*) = -1.312236, T_1(z_1(x^*)) = -1.116456, T_2(z_2(x^*)) = -1.312236, T_3(z_3(x^*)) = -1.312236.$$

To test α -Pareto optimality of this T- α -Pareto optimal solution, we solve the following problem:

$$\begin{aligned} & \max \sum_{i=1}^3 \varepsilon_i \\ & (0.875x_1 - 1.3x_2) - \varepsilon_1 \geq -1.116456, \\ & -(4.7x_1 + x_2) - \varepsilon_2 \geq -1.3122363, \\ & (6.15x_1 - 2x_2) - \varepsilon_3 \geq -1.3122363, \\ & 5x_1 + 7x_2 \leq 12, 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, x_1 + x_2 \geq 1, \\ & x_1, x_2, \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0. \end{aligned}$$

Using Lingo 15.0.32, we obtain the α -Pareto optimal solution $\bar{x}$ and $\bar{\varepsilon}$ as

$$\begin{aligned} \bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 &= 0, \bar{\lambda} = -1.312236, \\ \bar{x}_1 &= 0.08438819, \bar{x}_2 = 0.9156118, z_1(\bar{x}) = -1.116456, z_2(\bar{x}) = 1.312236, z_3(\bar{x}) = -1.312236, \\ T_1(z_1(\bar{x})) &= -1.116456, T_2(z_2(\bar{x})) = -1.312236, T_3(z_3(\bar{x})) = -1.312236. \end{aligned}$$

Here all $\bar{\varepsilon}_i = 0 \forall i = 1, 2, 3$. Hence the solution obtained in first stage x^* is itself α -Pareto optimal solution to the α -MOLPP.

Therefore usage of membership function or redefined membership function gives no feasible solution to the problem. But if we use well defined parametric T-characteristic functions, we may find α -Pareto optimal solutions. If we compare results obtained by using first set of proposed parametric T-characteristic functions with usual membership functions, we observe that the membership grade of $z_1(x)$ is more than 0.5; certainly much more acceptable scenario than infeasible solution to some problem. Moreover second set of parametric T-characteristic functions gives another α -Pareto optimal solution where conventional membership grade of objective function $z_2(x)$ is 1, and redefined membership grade is 1.12.

TABLE III

STATUS OF SOLUTIONS IN DIFFERENT CASES

Parametric T-characteristic Function	Status
--------------------------------------	--------

	of Solution
First set: Analogous to usual membership functions	No feasible solution
Second set: Analogous to redefined membership functions	No feasible solution
Third set: Usage of strictly monotonic part of membership function over entire domain	α – Pareto optimal solution exists
Fourth set: Usage of any strictly monotonic linear function	α – Pareto optimal solution exists

B. Numerical Example 2:

We consider another multiobjective linear programming problem *on investments made by private bank*, based on paper by J. L. Eatman, and C. W. Sealey Jr. [8]. To match with reality, we may assume that DM has imprecise goal for each of three objectives: a) maximization of profit, b) minimization of capital adequacy ratio, and c) minimization of illiquid risk assets. Also suppose that some coefficients viz. $C_{15}, C_{16}, C_{25}, C_{28}, C_{31}$ are triangular fuzzy numbers. So the MOLPP now becomes MOLPP with fuzzy coefficients under imprecise environment.

Then linear programming problem may be formulated as follows:

$$\begin{aligned}
 & \text{imprecise max } 0.04x_2 + 0.045x_3 + 0.055x_4 + C_{15}x_5 + C_{16}x_6 + 0.085x_7 + 0.092x_8 \\
 & \text{imprecise min } \frac{1}{20}(0.005x_2 + 0.04x_3 + 0.05x_4 + C_{25}x_5 + 0.1x_6 + 0.1x_7 + C_{28}x_8) \\
 & \text{imprecise min } \frac{1}{20}(C_{31}x_6 + x_7 + x_8) \\
 & \text{subject to} \\
 & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = (20 + 150 + 80), \\
 & x_1 \geq (0.14 \times 150) + (0.04 \times 80), \\
 & x_1 + 0.995x_2 + 0.96x_3 + 0.9x_4 + 0.85x_5 \geq (0.47 \times 150) + (0.36 \times 80), \\
 & x_j \geq 0.05(20 + 150 + 80), \quad \forall j = 1 \dots 8, \\
 & x_8 \geq 0.3(20 + 150 + 80), x_1 \dots x_8 \geq 0
 \end{aligned}$$

Here $C_{15}, C_{16}, C_{25}, C_{28}, C_{31}$ are all fuzzy coefficients in the form of triangular fuzzy numbers as

$$C_{15} = (0.06, 0.07, 0.08), C_{16} = (0.10, 0.105, 0.110), C_{25} = (0.070, 0.075, 0.080), C_{28} = (0.095, 0.10, 0.105), C_{31} = (0.8, 1.0, 1.2).$$

We suppose that the DM supplies aspiration level $\alpha = 0.7$ for these fuzzy coefficients.

1) First set of parametric T-characteristic functions:

Here we take T-characteristic functions for each of imprecise objective functions analogous to membership functions of fuzzy set theory. And each of the parametric T-characteristic functions is strictly monotonic over entire domain and higher membership grade of parametric T-characteristic functions give better result for objectives. Accordingly, we need to determine the goals, and tolerances of each of the imprecise objective functions. Therefore we compute individual maximum, and minimum of each of the imprecise objective functions under the given constraints and obtain the results as follows:

TABLE IV

INDIVIDUAL MAXIMUM, AND MINIMUM

Objective Function	Individual	Individual
	Maximum	Minimum
$z_1(x) = c_1x$	18.89438	11.95625
$z_2(x) = c_2x$	0.9338129	0.5996875
$z_3(x) = c_3x$	7.313125	4.9625

Next, suppose that the DM specifies the goal, and tolerance for objective function c_1x only. For other objective functions, we use individual maximum, and minimum values of objectives from table no 4 and construct parametric T-characteristic functions $T_i(c_ix), i = 1, 2, 3$, analogous to membership functions of fuzzy set theory as follows:

$$T_1(c_1x) = \frac{c_1x - 14}{2} \forall x \in X, T_2(c_2x) = \frac{0.9338129 - c_2x}{0.3341254} \forall x \in X, T_3(c_3x) = \frac{7.313125 - c_3x}{2.350625} \forall x \in X.$$

Based on these parametric T-characteristic functions, we convert the α -MOLPP into single objective linear programming problem [5] as follows:

$$\begin{aligned} & \max \lambda \\ & \text{subject to} \\ & \frac{1}{2}(0.04x_2 + 0.045x_3 + 0.055x_4 + 0.073x_5 + 0.1065x_6 \\ & \quad + 0.085x_7 + 0.092x_8 - 14) \geq \lambda, \\ & \frac{1}{0.3341254}\{0.9338129 - 0.05(0.005x_2 + 0.04x_3 + 0.05x_4 \\ & \quad + 0.0735x_5 + 0.1x_6 + 0.1x_7 + 0.0985x_8)\} \geq \lambda, \\ & \frac{1}{2.35}\{7.313125 - 0.05(0.94x_6 + x_7 + x_8)\} \geq \lambda, \end{aligned}$$

$$\begin{aligned}
 & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = (20 + 150 + 80), \\
 & x_1 \geq (0.14 \times 150) + (0.04 \times 80), x_j \geq 0.05(20 + 150 + 80), \text{ for } j = 1 \dots 8, \\
 & x_1 + 0.995x_2 + 0.96x_3 + 0.9x_4 + 0.85x_5 \geq (0.47 \times 150) + (0.36 \times 80), \\
 & x_8 \geq 0.3(20 + 150 + 80), \lambda \text{ is unrestricted in sign}, x_1 \dots x_8 \geq 0.
 \end{aligned}$$

Using Lingo 15.0.32, we obtain the T- α -Pareto optimal solution as

$$\begin{aligned}
 & \lambda^* = 0.81, x_1^* = 24.2, x_2^* = 78.84, x_3^* = x_4^* = x_5^* = x_7^* = 12.5, x_6^* = 21.96, x_8^* = 75, z_1(x^*) = 15.62, z_2(x^*) = 0.66, \\
 & z_3(x^*) = 5.41, T_1(z_1(x^*)) = T_2(z_2(x^*)) = 0.81, T_3(z_3(x^*)) = 0.811.
 \end{aligned}$$

To test α -Pareto optimality of this T- α -Pareto optimal solution, we solve the following problem

$$\begin{aligned}
 & \max \sum_{i=1}^3 \varepsilon_i \\
 & \text{subject to} \\
 & \frac{1}{2}(0.04x_2 + 0.045x_3 + 0.055x_4 + 0.073x_5 + 0.1065x_6 + 0.085x_7 + 0.092x_8 - 14) - \varepsilon_1 \geq 0.8087605, \\
 & \frac{1}{0.3341254} \{ (0.9338129 - 0.05(0.005x_2 + 0.04x_3 + 0.05x_4 + 0.0735x_5 + 0.1x_6 + 0.1x_7 + 0.0985x_8)) \} - \varepsilon_2 \geq 0.8087605, \\
 & \frac{1}{2.350625} \{ 7.313125 - 0.05(0.94x_6 + x_7 + x_8) \} - \varepsilon_3 \geq 0.8107958, \\
 & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = (20 + 150 + 80), x_1 \geq (0.14 \times 150) + (0.04 \times 80), \\
 & x_1 + 0.995x_2 + 0.96x_3 + 0.9x_4 + 0.85x_5 \geq (0.47 \times 150) + (0.36 \times 80), \\
 & x_j \geq 0.05(20 + 150 + 80), \text{ for } j = 1, 2 \dots 8, x_8 \geq 0.3(20 + 150 + 80), x_1 \dots x_8, \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0.
 \end{aligned}$$

Using Lingo 15.0.32 to solve this problem, the α -Pareto optimal solutions $\bar{x}$ and $\bar{\varepsilon}$ are as follows:

$$\begin{aligned}
 & \bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 = 0, \bar{\lambda} = 0.81, \bar{x}_1 = 24.2, \bar{x}_2 = 78.84, \bar{x}_3 = \bar{x}_4 = \bar{x}_5 = \bar{x}_7 = 12.5, \bar{x}_6 = 21.96, \bar{x}_8 = 75, \\
 & z_1(\bar{x}) = 15.62, z_2(\bar{x}) = 0.66, z_3(\bar{x}) = 5.41, T_1(z_1(\bar{x})) = T_2(z_2(\bar{x})) = 0.81, T_3(z_3(\bar{x})) = 0.811.
 \end{aligned}$$

Here all $\bar{\varepsilon}_i = 0 \forall i = 1, 2, 3$. Hence the solution obtained in first stage i.e. x^* is itself α -Pareto optimal solution to the α -MOLPP.

2) Second set of parametric T-characteristic functions:

Instead of using parametric T-characteristic function analogous to membership function of fuzzy set theory, we construct another set of parametric T-characteristic functions to obtain α -Pareto optimal solution of α -MOLPP for fixed $\alpha=0.7$ as follows (so that each of the parametric T-characteristic functions are strictly monotonic over entire domain, and higher membership grade of parametric T-characteristic functions give better result for objectives)

$$T_1(c_1x) = 0.04x_2 + 0.045x_3 + 0.055x_4 + 0.073x_5 + 0.1065x_6 + 0.085x_7 + 0.092x_8, \forall x \in X$$

$$T_2(c_2x) = -\frac{1}{20}(0.005x_2 + 0.04x_3 + 0.05x_4 + 0.0735x_5 + 0.1x_6 + 0.1x_7 + 0.0985x_8), \forall x \in X$$

$$T_3(c_3x) = -\frac{1}{20}(0.94x_6 + x_7 + x_8), \forall x \in X$$

Based on these parametric T-characteristic functions, we now convert the α -MOLPP into single objective linear programming problem [6] as follows

$$\begin{aligned} & \max \lambda \\ & \text{subject to} \\ & 0.04x_2 + 0.045x_3 + 0.055x_4 + 0.073x_5 + 0.1065x_6 + 0.085x_7 + 0.092x_8 \geq \lambda, \\ & -0.05(0.005x_2 + 0.04x_3 + 0.05x_4 + 0.0735x_5 + 0.1x_6 + 0.1x_7 + 0.0985x_8) \geq \lambda, \\ & -0.05(0.94x_6 + x_7 + x_8) \geq \lambda, \\ & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = (20 + 150 + 80), \\ & x_1 \geq (0.14 \times 150) + (0.04 \times 80), \\ & x_1 + 0.995x_2 + 0.96x_3 + 0.9x_4 + 0.85x_5 \geq (0.47 \times 150) + (0.36 \times 80), \\ & x_j \geq 0.05(20 + 150 + 80), \text{ for } j=1 \dots 8, \\ & x_8 \geq 0.3(20 + 150 + 80), \\ & x_1 \dots x_8 \geq 0, \lambda \text{ is unrestricted in sign.} \end{aligned}$$

Using Lingo 15.0.32, we obtain the T- α -Pareto optimal solution as

$$\begin{aligned} & \lambda^* = -4.96, x_1^* = 24.2, x_2^* = 88.3, x_3^* = x_4^* = x_5^* = x_6^* = x_7^* = 12.5, x_8^* = 75, \\ & z_1(x^*) = 14.99, z_2(x^*) = 0.62, z_3(x^*) = 4.96, T_1(z_1(x^*)) = 14.99, T_2(z_2(x^*)) = -0.62, T_3(z_3(x^*)) = -4.96. \end{aligned}$$

To test α -Pareto optimality of this T- α -Pareto optimal solution, we solve the following problem

$$\begin{aligned}
 & \max \sum_{i=1}^3 \varepsilon_i \\
 & \text{subject to} \\
 & 0.04x_2 + 0.045x_3 + 0.055x_4 + 0.073x_5 + 0.1065x_6 + 0.085x_7 + 0.092x_8 - \varepsilon_1 \geq 14.98825, \\
 & -0.05(0.005x_2 + 0.04x_3 + 0.05x_4 + 0.0735x_5 + 0.1x_6 + 0.1x_7 + 0.0985x_8 - \varepsilon_2) \geq -0.6186375, \\
 & -0.05(0.94x_6 + x_7 + x_8) - \varepsilon_3 \geq -4.9625, \\
 & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = (20 + 150 + 80), \\
 & x_1 \geq (0.14 \times 150) + (0.04 \times 80), \\
 & x_1 + 0.995x_2 + 0.96x_3 + 0.9x_4 + 0.85x_5 \geq (0.47 \times 150) + (0.36 \times 80), \\
 & x_j \geq 0.05(20 + 150 + 80), \text{ for } j = 1 \dots 8, \\
 & x_8 \geq 0.3(20 + 150 + 80), x_1 \dots x_8, \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0.
 \end{aligned}$$

Using Lingo 15.0.32 to solve this problem, we obtain the α -Pareto optimal solution $\bar{x}$ and $\bar{\varepsilon}$ as

$$\begin{aligned}
 & \bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 = 0, \bar{\lambda} = -4.96, \bar{x}_1 = 24.2, \bar{x}_2 = 88.3, \bar{x}_3 = \bar{x}_4 = \bar{x}_5 = \bar{x}_6 = \bar{x}_7 = 12.5, \bar{x}_8 = 75, \\
 & z_1(\bar{x}) = 14.99, z_2(\bar{x}) = 0.62, z_3(\bar{x}) = 4.96, T_1(z_1(\bar{x})) = 14.99, T_2(z_2(\bar{x})) = -0.62, T_3(z_3(\bar{x})) = -4.96.
 \end{aligned}$$

Here all $\bar{\varepsilon}_i = 0 \forall i = 1, 2, 3$. Hence the solution obtained in first stage x^* is itself α -Pareto optimal solution to the α -MOLPP for fixed α .

TABLE V
DIFFERENT PARETO OPTIMAL SOLUTIONS

Objective Functions	Optimal Values Using	
	First Set of Parametric	Second Set of Parametric
	T-characteristic Functions	T-characteristic Functions

Profit	15.62	14.99
Capital-adequacy Ratio	0.66	0.62
Illiquid Risk Assets	5.41	4.96

Clearly first set of parametric T-characteristic functions in α -MOLPP results in higher profit whereas second set of parametric T-characteristic functions gives better result for capital-adequacy ratio as well as illiquid risk assets, two important pillars for optimal decision of this example. And both are α -Pareto optimal solutions.

VII. CONCLUSIONS

We have observed that Wu *et al* (2015) [17] did not use redefined membership function according to their proposed definition. In numerical example one of this paper, we have shown that although redefined membership function in α -MOLPP correctly identifies the best α -Pareto optimal solution among the good cases; it fails to identify bad α -Pareto optimal solution among the worst cases! And proper usage of membership or even redefined membership functions may suggest an α -MOLPP infeasible, although we may find α -Pareto optimal solution(s) by using parametric T-characteristic functions.

On the other hand, in real life scenarios, DM may not be able to identify the goal or tolerance or both; and in some cases, DM may not be present at all. In these cases, it is difficult to construct membership function or redefined membership function in imprecise environment. And the usage of well-defined parametric T-characteristic function decreases frequent interaction with DM [11], and thus it saves precious time of both.

Further higher value of membership function generating better value for objective is one of the primary aims of using membership function in imprecise environment. And if we use usual membership function or redefined membership function to solve MOLPP with fuzzy coefficients, we can find at most one α –Pareto optimal solution in imprecise environment.

But if we use parametric T-characteristic functions, we may find different solutions that are all α –Pareto optimal. Finally DM is the king, and his choice is always final. And scientific, and especially optimization tools must try to supply more than one ‘best choice’ to DM, if possible; and that should hold even more in imprecise environment. So usage of parametric T-characteristic function in solving MOLPP with fuzzy coefficients is more suitable to DM than membership or redefined membership function under fuzzy environment.

Hence in α –MOLPP under imprecise environment, we propose to take general procedure of using parametric T-characteristic functions covering the entire set of possible solutions in place of membership functions and redefined membership functions. From all these we may add that under these circumstances the issue of infeasibility in α –MOLPP even by using fuzzy set theory yielded from several published methods seems worthwhile or even necessary to reconsider.

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